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LETTER TO THE EDITOR

Invariance transformations and the Hénon-Heiles problem

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Abstract. The infinitesimal symmetry transformations associated with the Hénon-Heiles problem are determined. It is shown that the invariance group is a one-parameter group and the corresponding first integral is the energy. The method used indicates that no other symmetry transformations or exact integral exist.

Hénon and Heiles (1964) studied the bounded motion of orbits for a system governed by the Hamiltonian

$$H = \tfrac{1}{2}(p_1^2 + p_2^2 + q_1^2 + q_2^2) + q_1^2 q_2 - \tfrac{1}{3}q_2^2 \qquad (1)$$

($p_1 = \dot{q}_1, p_2 = \dot{q}_2$). For this system the total energy is a constant of the motion. Hence the trajectories move in a four-dimensional phase space but are restricted to a three-dimensional surface. It is then possible to study a two-dimensional cross section of the three-dimensional energy surface. For example, we can plot successive points of the trajectory ($q_2 = 0, p_2 > 0$) in the (q_1, p_1) plane. If the only constant of motion is the total energy, then the points should be free to wander through the region of the (q_1, p_1) plane corresponding to the energy surface. If there is an additional constant of the motion the points will lie on a smooth curve in the (q_1, p_1) plane. For sufficiently small values of the energy, Hénon and Heiles found only smooth curves, indicating that to computer accuracy there was a first integral independent of the energy. Their numerical results have led to other studies (Lunsford and Ford 1972) in order to understand the motion and to find this additional constant of the motion (Leach 1981). However the progress has not been great and the term 'notorious' has been applied to the Hénon-Heiles problem as an indication of the sense of frustration produced.

From the standpoint of group theory the Hénon-Heiles problem has been analysed by considering the infinitesimal generators in the Lie theory of extended groups (Leach 1981). In the present letter we consider another group theoretical approach. Our method is based on determining Lie algebras associated with a given Newtonian equation (Aguirre and Krause 1984); but, unlike the method used by Leach, in our approach it is not necessary to know the explicit expressions of the Lie group generators. This approach has been proposed originally by Aguirre and Krause (1984, 1985) in the study of linear and non-linear Newtonian systems and extended by Soares Neto and Vianna (1988) for the study of third-order ordinary differential equations. In the Hénon-Heiles problem, as we shall see, the basic equations and the binding relations

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of our method have an immediate solution; as a consequence the system of differential equations involving the Lie algebra structure constants is not necessary to determine the invariance group.

We formulate the problem by considering the equations of the motion derived from Hamiltonian (1). They have the general form ($a, b, c, f \neq 0$)

$$\ddot{q}_1 = aq_1 + bq_1q_2 \tag{2}$$

$$\ddot{q}_2 = cq_2 + fq_1^2 \tag{3}$$

where in the Hénon-Heiles problem the constants a, b, c, f are $a = f = -1$, $b = -2$, $c = -\frac{1}{3}$.

We consider the symmetries generated by infinitesimal transformations of the form

$$t' = t + \varepsilon N(q_1, q_2, t) \tag{4}$$

$$q_1' = q_1 + \varepsilon T_1(q_1, q_2, t) \tag{5}$$

$$q_2' = q_2 + \varepsilon T_2(q_1, q_2, t) \tag{6}$$

where ε denotes a parameter such that $0 < \varepsilon \ll 1$.

Assuming the invariance of (2) and (3) under infinitesimal transformations (4), (5) and (6), we obtain that

$$\partial_{11}N = \partial_{22}N = \partial_{t1}N = \partial_{t2}N = \partial_{12}N = 0$$

$$\partial_{11}T_1 = \partial_{22}T_1 = \partial_{12}T_1 = 0$$

$$\partial_{11}T_2 = \partial_{22}T_2 = \partial_{12}T_2 = 0$$

where

$$\partial_{ij} = \frac{\partial^2}{\partial q_i \partial q_j} \qquad \partial_{tj} = \frac{\partial^2}{\partial t \partial q_j} \qquad i, j = 1, 2.$$

Thus we can write

$$N(q_1, q_2, t) = C_1q_1 + C_2q_2 + \phi(t) \tag{7}$$

$$T_1(q_1, q_2, t) = \Omega_1(t)q_1 + \Omega_2(t)q_2 \tag{8}$$

$$T_2(q_1, q_2, t) = \Omega_3(t)q_1 + \Omega_4(t)q_2 \tag{9}$$

where C_1 and C_2 are constants.

On the other hand we find, also from the assumed invariance of (2) and (3), that

$$(3aq_1 + 3bq_1q_2)\partial_1N + (cq_2 + fq_1^2)\partial_2N + \partial_{tt}N - 2\partial_{t1}T_1 = 0 \tag{10}$$

$$(2aq_1 + 2bq_1q_2)\partial_2N - 2\partial_{t2}T_1 = 0 \tag{11}$$

$$(2cq_2 + 2fq_1^2)\partial_1N - 2\partial_{t1}T_2 = 0 \tag{12}$$

$$(3cq_2 + 3fq_1^2)\partial_2N + (aq_1 + bq_1q_2)\partial_1N + \partial_{tt}N - 2\partial_{t2}T_2 = 0 \tag{13}$$

$$(2cq_2 + 2fq_1^2)\partial_tN - (aq_1 + bq_1q_2)\partial_1T_2 - (cq_2 + fq_1^2)\partial_2T_2 - \partial_{tt}T_2 + cT_2 + 2fq_1T_1 = 0 \tag{14}$$

$$(2aq_1 + 2bq_1q_2)\partial_tN - (aq_1 + bq_1q_2)\partial_1T_1 - (cq_2 + fq_1^2)\partial_2T_1 - \partial_{tt}T_1 + (a + bq_2)T_1 + bq_1T_2 = 0 \tag{15}$$

with

$$\partial_i = \frac{\partial}{\partial q_i} \qquad \partial_t = \frac{\partial}{\partial t} \qquad \partial_{tt} = \frac{\partial^2}{\partial t^2}.$$

Substituting (7), (8) and (9) into (10)-(15) and separating out the terms which are q_i dependent, q_iq_j dependent and q_i free, we have

$$C_1 = C_2 = 0 \qquad \Omega_2(t) = \Omega_3(t) = 0 \qquad \Omega_4(t) = \Omega_1(t) \tag{16}$$

the basic equations

$$\ddot{\phi}(t) = 0 \tag{17}$$

$$\Omega_1(t) = 2\dot{\phi}(t) \tag{18}$$

and the binding relations

$$\dddot{\phi}(t) + 2\dot{\Omega}_1(t) = 0 \tag{19}$$

$$\ddot{\Omega}_1(t) - 2c\dot{\phi}(t) = 0 \tag{20}$$

$$\ddot{\Omega}_1(t) - 2a\dot{\phi}(t) = 0. \tag{21}$$

Equations (17) and (18) can be easily solved; a general solution of (17) is

$$\phi(t) = At + B \qquad \text{with } A \text{ and } B \text{ constants.} \tag{22}$$

So, from (18) we obtain

$$\Omega_1(t) = 2A \tag{23}$$

and from the binding relations, it follows that

$$A = 0. \tag{24}$$

Thus, for the infinitesimal symmetry transformations of (2) and (3), we have found that

$$N(q_1, q_2, t) = B \qquad T_1(q_1, q_2, t) = 0 \qquad T_2(q_1, q_2, t) = 0. \tag{25}$$

Hence the only infinitesimal generator for the Hénon-Heiles problem is

$$X = B\frac{\partial}{\partial t} \qquad B = \text{constant} \tag{26}$$

and the corresponding first integral is the energy.

In conclusion, we have shown that the invariance transformations of the system (2), (3) are such that

$$t' = t + \varepsilon B \qquad q_1' = q_1 \qquad q_2' = q_2.$$

The Hénon-Heiles problem concerns (2) and (3) for a particular choice of the constants a, b, c, f. It then follows that by our method the Hénon-Heiles problem has only one generator for a one-parameter group and the corresponding first integral is the energy. A similar result has been pointed out by Leach (1981) using the method of the Lie theory of extended groups. With the method used by us, in general, to determine the invariance group we must solve a system of differential equations involving the Lie algebra structure constants (see Soares Neto and Vianna 1988, for instance). In the case of (2) and (3), however, that system of differential equations has not been necessary; our results have been obtained directly from the invariance of (2) and (3) under transformations (4), (5) and (6). On the other hand, it is known (Lutzky 1978) that the complete symmetry group of a physical system is determined by considering the invariance of the equation of motion. Our result, therefore, indicates also that no other symmetry transformations or exact integral exist for the Hénon-Heiles problem.

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